Hideo Okawara's
Mixed Signal Lecture Series

DSP-Based Testing – Fundamentals 12
Spectrum Estimation

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Preface to the Series

ADC and DAC are the most typical mixed signal devices. In mixed signal testing, analog stimulus signal is generated by an arbitrary waveform generator (AWG) which employs a D/A converter inside, and analog signal is measured by a digitizer or a sampler which employs an A/D converter inside. The stimulus signal is created with mathematical method, and the measured signal is processed with mathematical method, extracting various parameters. It is based on digital signal processing (DSP) so that our test methodologies are often called DSP-based testing.

Test/application engineers in the mixed signal field should have thorough knowledge about DSP-based testing. FFT (Fast Fourier Transform) is the most powerful tool here. This corner will deliver a series of fundamental knowledge of DSP-based testing, especially FFT and its related topics. It will help test/application engineers comprehend what the DSP-based testing is and assorted techniques.

Editor's Note

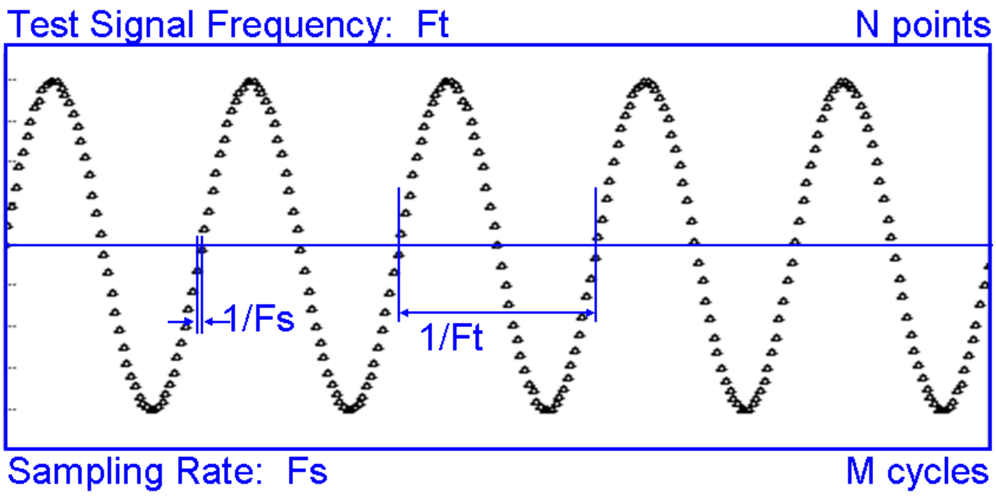
For other articles in this series, please visit the Verigy web site at www.verigy.com/go/gosemi.

Spectrum Estimation

In a previous go/semi article, windowing was discussed for coping with the situation that the measurement period does not contain an integer number of signal cycles. Two windows were introduced. FLAT_TOP window is useful and accurate in regular amplitude estimation without any special data manipulation. HANNING window may be less popular than FLAT_TOP in general. However, there is a remarkably marvelous application available with HANNING window. It was developed and reported by Dr. Tabei and Dr. Ueda in 1987. In this article, the author is going to digest the core part of the original paper, and show how to deploy it in practical procedure.

Review of Fractional M Situation

You have already seen the coherency equation many times in this series of articles for go/semi, and the coherent condition is shown in Figure 1.



Coherent Condition: $\dfrac{Ft}{Fs} = \dfrac{M}{N}$

Figure 1: Coherency Condition

M and N should be integer numbers and have no common divisors. In Figure 2 the yellow waveform and spectrum show a good coherency condition. The unit test period (UTP) exactly contains an integer number of test signal cycles. On the other hand the red line shows the situation that an integer number of signal cycles cannot be captured in the UTP for some reason. Consequently the spectrum gets severely smeared.

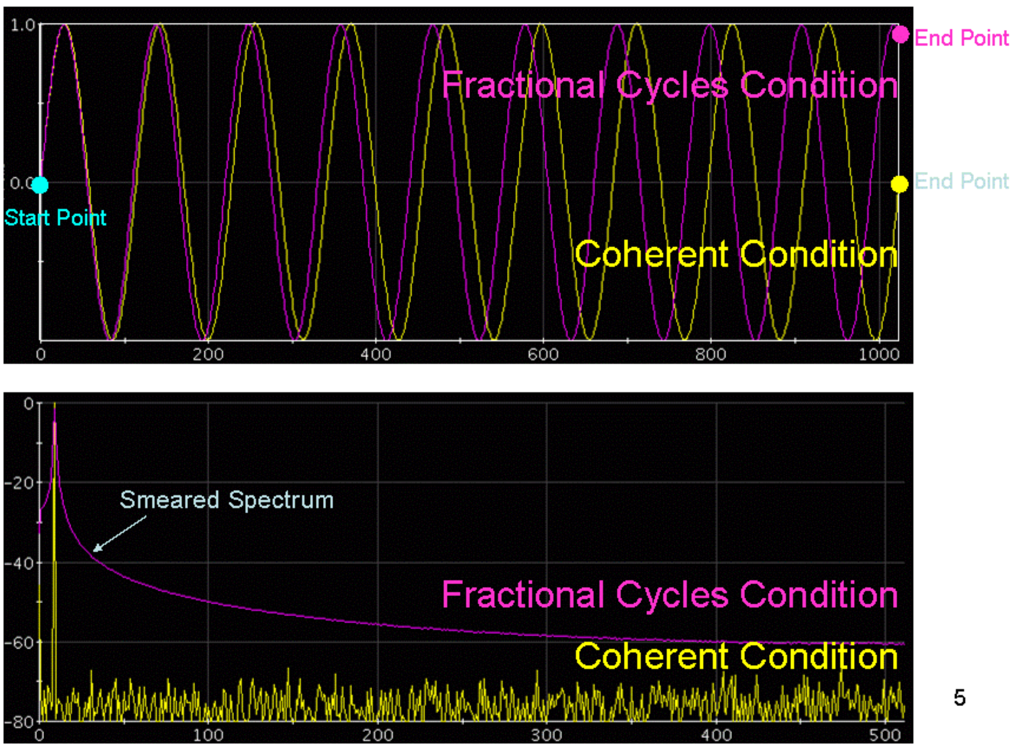


Figure 2: Smeared Spectrum

Tabei & Ueda Method

In reality, we would often have such situations that a UTP cannot capture a whole number of signal cycles, and we need to work around such cases in spectrum analyses. Windowing is one of the practical workarounds, and it contributes to improve measurement results to a certain extent. Dr. Tabei and Dr. Ueda devised an elegant interpolation method to accurately estimate the frequency, amplitude and phase of the test signal with applying FFT and Hanning window. The method was introduced in the paper below.

Makoto TABEI and Mitsuhiro UEDA, "A Method of High Precision Frequency Detection with FFT", IEIEC Transaction A Vol. J70-A No.5 pp.798-805, May 1987

This is so valuable method for our test cases that the most important part of it is digested as follows.

FFT Response and Frequency Calculation

The test signal is continuous analog signal supposed to contain no noise in this section. The A/D converted data can be written as follows;

$$x(i) = Ae^{j2\pi\left(P_o + \frac{if}{N}\right)} \qquad (i=0,\ 1,\ 2,\ \ldots,\ N-2,\ N-1) \qquad (1)$$

where
N: Length of the input data (Number of points)
f: Actual Signal Frequency
A: Amplitude
Po: Initial Phase

If the signal is real, it is supposed to be the sum of positive and negative frequency signals as the equation below.

$$\cos(t) = \frac{1}{2}\left(e^{jt} + e^{-jt}\right) \qquad (2)$$

So it is reasonable for the input signal to be expressed as complex form.

Supposing that the signal does not contain a whole number of signal cycles in the unit test period, let's employ the Hanning window. The Hanning window of the length N can be defined as follows;

$$w(i) = 1 - \cos\left(\frac{2\pi i}{N}\right) \qquad (\mathbf{i}=0,\ 1,\ \ldots,\ \mathbf{N}-2,\ \mathbf{N}-1) \qquad (3)$$

With multiplying the Hanning window **w**(**i**) to the input signal **x**(**i**), the discrete Fourier transform **G**(**k**) can be expressed as follows;

$$G(k) = \sum_{i=0}^{N-1}\left\{w(i)\cdot x(i)e^{-j\frac{2\pi ik}{N}}\right\} = \sum_{i=0}^{N-1}\left\{\left(1-\cos\left(\frac{2\pi i}{N}\right)\right)Ae^{j2\pi\left(P_o+\frac{fi}{N}\right)}e^{-j\frac{2\pi ik}{N}}\right\}$$

$$(\mathbf{k}=0,\ 1,\ \ldots,\ \mathbf{N}-2,\ \mathbf{N}-1) \qquad (4)$$

With considering Equation (2), the Hanning window can be modified to exponential expression. Then Equation (4) is modified as follows;

$$G(k) = Ae^{j2\pi P_o}\sum_{i=0}^{N-1}\left\{e^{j\frac{2\pi(f-k)i}{N}} - \frac{1}{2}e^{j\frac{2\pi(f-k+1)i}{N}} - \frac{1}{2}e^{j\frac{2\pi(f-k-1)i}{N}}\right\}$$

(**k**=0, 1, ..., **N**-2, **N**-1) (5)

The sum of geometric series with the common ratio **a** can be described as follows.

$$\sum_{i=0}^{N-1} a^i = \frac{a^N - 1}{a-1} \qquad (a \neq 1) \tag{6}$$

Another equation is available.

$$e^{jt} - 1 = \left(e^{j\frac{t}{2}} - e^{-j\frac{t}{2}}\right)e^{j\frac{t}{2}} = 2j\sin\left(\frac{t}{2}\right)e^{j\frac{t}{2}} \tag{7}$$

With considering (6) and (7), Equation (5) can be modified as follows;

$$G(k) = Ae^{j2\pi P_o}\frac{-\cos\frac{\pi(f-k)}{N}\sin^2\frac{\pi}{N}}{\sin\frac{\pi(f-k-1)}{N}\sin\frac{\pi(f-k)}{N}\sin\frac{\pi(f-k+1)}{N}}\sin(\pi(f-k))e^{j\pi(f-k)}$$

(**k**=0, 1, ... , **N**-2, **N**-1) (8)

This is the response to the input signal with Hanning window. When no window is applied, the similar processing derives the result **H**(**k**) which is well-known response function of discrete Fourier transform. **H**(**k**) is derived as follows;

$$H(k) = Ae^{j2\pi P_o}\frac{e^{-j\frac{\pi(f-k)}{N}}}{\sin\frac{\pi(f-k)}{N}}\sin(\pi(f-k))e^{j\pi(f-k)}$$ (**k**=0, 1, ... , **N**-2, **N**-1) (9)

If N is large enough, the response functions of **G**(**k**) and **H**(**k**) are shown in Figure A. In |**H**(**k**)|, the local maximum of the side lobe decreases as proportional to $|\mathbf{f}\text{-}\mathbf{k}|^{-1}$, while the main lobe width of |**G**(**k**)| is twice larger than |**H**(**k**)|. Consequently it makes easy to interpolate in the vicinity of the maximum. On the other hand, the local maximum of the side lobe of |**G**(**k**)| rapidly decreases as proportional to $|\mathbf{f}\text{-}\mathbf{k}|^{-3}$ and converges to almost 0. If |**f**-**k**|>3, it can be regarded as 0. Consequently if other frequency components in the input series are far away (>3 bins) from **f**, they would give no impact to the response around **f.**

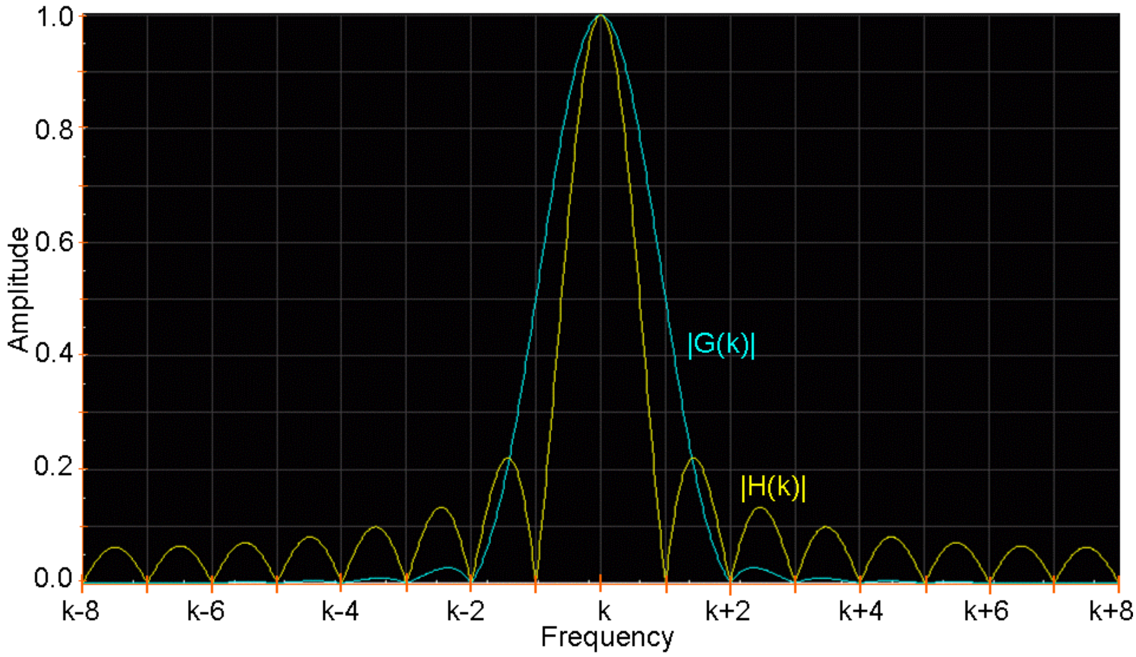


Figure A: Response of DFT

Let's define the suffix **k** of the maximum elements of |**G**(**k**)| as **kmax**. Figure B shows the responses of |**G**(**kmax**-1)|, |**G**(**kmax**)| and |**G**(**kmax**+1)|. If **f** can be limited as below;

$$k_{max} - \frac{1}{2} \le f \le k_{max} + \frac{1}{2} \tag{11}$$

You can estimate **f** with using |**G**(**kmax**-1)|, |**G**(**kmax**)| and |**G**(**kmax**+1)|.

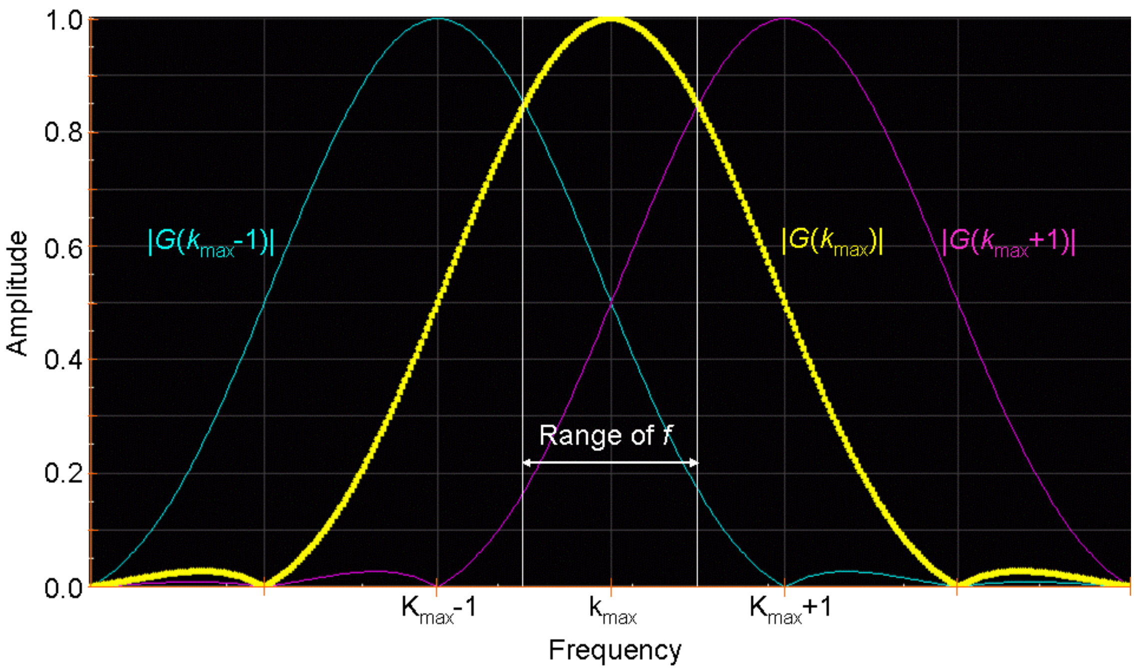


Figure B: Response of maximum output element

Let's introduce **r** as the ratio of **G**(**kmax**-1) and **G**(**kmax**) with considering Equation (8) as follows.

$$r=\frac{\left|G(k_{\max}-1)\right|}{\left|G(k_{\max})\right|}=\frac{\left|\cos\left(\frac{\pi(f-k_{\max}+1)}{N}\right)\sin\left(\frac{\pi(f-k_{\max}-1)}{N}\right)\right|}{\left|\cos\left(\frac{\pi(f-k_{\max})}{N}\right)\sin\left(\frac{\pi(f-k_{\max}+2)}{N}\right)\right|} \tag{12}$$

If **N** is large enough, the following approximations can be possible.

$$\cos(\theta)\cong 1 \qquad (|\theta|<<1) \tag{13}$$

$$\sin(\theta)\cong \theta \qquad (|\theta|<<1) \tag{14}$$

Equation (12) can be simplified with considering Equations (13) and (14) as follows.

$$r\cong\frac{\left|\frac{\pi(f-k_{\max}-1)}{N}\right|}{\left|\frac{\pi(f-k_{\max}+2)}{N}\right|}=-\frac{f-k_{\max}-1}{f-k_{\max}+2} \tag{15}$$

Consequently **f** can be derived as follows.

$$f=k_{\max}+\frac{1-2r}{1+r} \tag{16}$$

This is the one result of the interpolation equations. The approximation of Equations (13) and (14) has the effect that the error in **r** is less than 1% at **N**=32. When N becomes larger, the error decreases as proportional to $\mathbf{N}^{-2}$. For the infinite **N**, Equation (16) is strictly true. In general **N** is much greater than 32 so that Equation (16) delivers a very good approximation.

In the similar manner, let's introduce **s** as the ratio of **G**(**kmax**+1) and **G**(**kmax**).

$$s=\frac{\left|G(k_{\max}+1)\right|}{\left|G(k_{\max})\right|} \tag{17}$$

$$f=k_{\max}-\frac{1-2s}{1+s} \tag{18}$$

Equations (16) and (18) can be consistent at the same time, and either one can estimate **f**.

In reality, **G**(**k**) is the measurement result so that it includes noise. Therefore the equation less sensitive to noise is more useful. Let's differentiate Equation (16) by **r**.

$$\frac{df}{dr}=\frac{-3}{(1+r)^2} \tag{19}$$

Equation (19) tells that the larger **r** is, the smaller the absolute value of the derivative is. i.e. the greater |**G**(**kmax**-1)|/|**G**(**kmax**)| is, the smaller the absolute value of the derivative is. The same discussion goes with |**G**(**kmax**+1)|. Therefore when |**G**(**kmax**-1)| is greater than |**G**(**kmax**+1)|, use Equation (16). Otherwise, use Equation (18). Then the effect of the noise in **G**(**kmax**-1), **G**(**kmax**) and **G**(**kmax**+1) can be minimized.

The amplitude **A** and the initial phase **Po** can be resolved by Equation (8) with substituting **f** and **kmax**. You can calculate them as they are. However, the signal can be simplified as follows, applying the approximation of Equations (13) and (14);

$$Ae^{j2\pi P_o} = -\frac{G(k_{\max})}{N}\frac{\pi(f-k_{\max})}{\sin(\pi(f-k_{\max}))}\times(f-k_{\max}-1)(f-k_{\max}+1)e^{-j\pi(f-k_{\max})} \quad (20)$$

Finally the amplitude and the phase can be derived as follows.

$$A = \frac{-|G(k_{\max})|}{N}\frac{\pi(f-k_{\max})}{\sin(\pi(f-k_{\max}))}\times(f-k_{\max}-1)(f-k_{\max}+1) \quad (21)$$

$$P_o = \frac{1}{2\pi}Arg\left(G(k_{\max})e^{-j\pi(f-k_{\max})}\right) \quad (22)$$

Now that you can see that discrete Fourier transform array **G**(**k**) which is applied Hanning window can estimate **f**, **A** and **Po** with Equations (12), (16), (17), (18), (21) and (22). The discussion above assumes that for simplicity there is a single frequency component in the input data. However, the response in Figure A shows that when there are two signals, if they are away from each other by more than 4 bins, there is almost no interference between them. Therefore, even if there would be multiple big spectral components contained in an input data, you could estimate each one of them, if they are not too close with each other. In this case, the maximum spectrum should be considered as the local maximum one.

Readers of this article understand how the interpolation method is derived. The rest of the paper talks about noise and accuracy, summarizing as guidance to apply it correctly as follows.

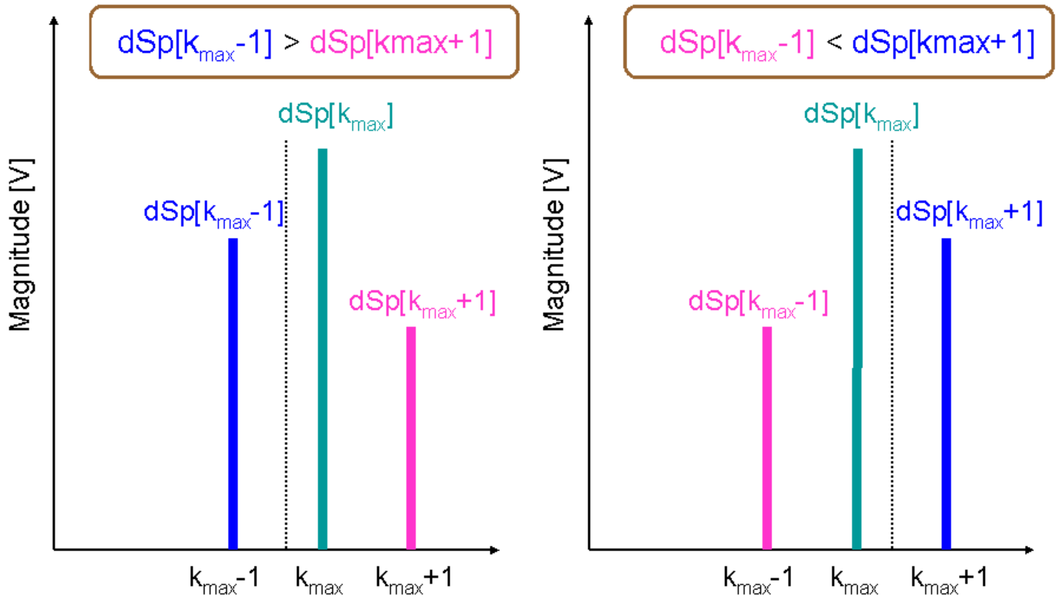
Guidance

Noise Effect:
The noisier the input signal is, the less accurate the estimation is.
Roughly 1% error @20dB SNR, and 0.1% error @40dB SNR

Conditioning:
The greater **N** is, the better accuracy is obviously. (**N** should be greater than 256.)
Avoid near DC (bin# <5) and near Nyquist (bin# >(Nyquist-5)) for accuracy.
Interference tones should be more than 5 bins away from the target signal.

Actual Programming

In the actual measurement, firstly apply DSP_FFT( ) with HANNING window. Then search the local maximum bin (**kmax**) and the second maximum bin between **kmax**-1 and **kmax**+1. As Figure 3 depicts, the ratio (**dR**) of the 2nd maximum bin and the 1st maximum bin becomes the key parameter.



$$dR = \frac{dSp[k_{max} - 1]}{dSp[k_{max}]} \qquad dR = \frac{dSp[k_{max} + 1]}{dSp[k_{max}]}$$

Figure 3: Search Local Maximum and 2nd Maximum

The model program for a single tone measurement would be as follows.

```
INT                  i,kmax,Ndgt,Nsp;
DOUBLE               dR,dX,dY,dMax,dFresln;
DOUBLE               dFrequency,dAmplitued,dPhase;
COMPLEX              CX,CY,CZ;
ARRAY_D              dVdgt,dSp,dTemp;
ARRAY_COMPLEX        CSp;

dVdgt=DGT("Aout").getWaveform();
dFdgt=1.0/DGT("Aout").getConvClock();
Ndgt=dVdgt.size();
dFresln=dFdgt/Ndgt; // Frequency resolution(bin space)

DSP_FFT(dVdgt,CSp,HANNING);    //  Apply HANNING window
Nsp=CSp.size();
DSP_RECT_POL(CSp,dSp,dTemp);   // Spectrum length

dMax=0.0;  kmax=0;
for (i=0;i<Nsp;i++) {        // Search the maximum length
    if (dSp[i]>dMax) { dMax=dSp[i]; kmax=i; }
}

if (dSp[kmax-1]>dSp[kmax+1]) {
    dR=dSp[kmax-1]/dSp[kmax];              // Equation(12)
    dFx=kmax+(1.0-2.0*dR)/(1.0+dR);      // Equation(16)
} else {
    dR=dSp[kmax+1]/dSp[kmax];              // Equation(17)
    dFx=kmax-(1.0-2.0*dR)/(1.0+dR);      // Equation(18)
}
dFrequency=dFx*dFresln;            // Estimated Frequency
                                   // dFx: Estimated Bin Location
dX=dFx-kmax;
```

```
32:    dY=M_PI*dX;
33:    dAmplitude=-dSp[kmax]*(dY/sin(dY))*(dX-1.0)*(dX+1.0);
34:                                                  // Equation(21)
35:    CX=CSp[kmax];
36:    CY.real()= cos(dY);
37:    CY.imag()=-sin(dY);
38:    CZ=CX*CY;
39:    dPhase=atan2(CZ.imag(),CZ.real());     // Equ.(22)[rad]
40:
```

List 1: Source Code Example for Single Tone Measurement

In the original paper, Equation (21) contains 1/**N,** which is a scaling factor in Fourier transform. DSP_FFT() delivers **G**(**k**)/**N** so that the line 33 has no 1/**N**. If you need the frequency estimation only, you need not the lines 30 and later. If you need the amplitude only, you need not the lines 34 and later. If you need to estimate multiple tones, the search range in Line 18 should be appropriately modified and narrowed. This would be adjusted during the online debug.

Example

This is a simulation to understand how the Tabei & Ueda interpolation method performs. A four-tone signal is supposed to be digitized 4096 points by an 8-bit ADC at the sampling rate of 8kHz. Actually the test signal is a harmony of chord "G7." Figure 4 shows the captured wave-form. The waveform contains 8-bit quantization noise so that SNR would be approximately 50dB.

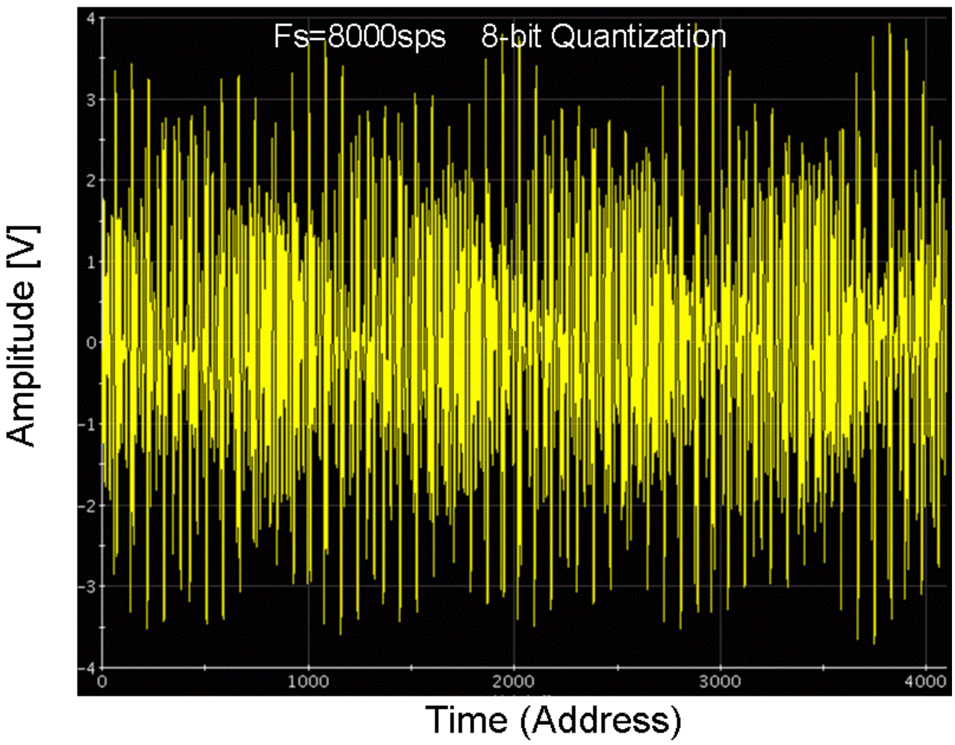


Figure 4: Captured Waveform (Simulation)

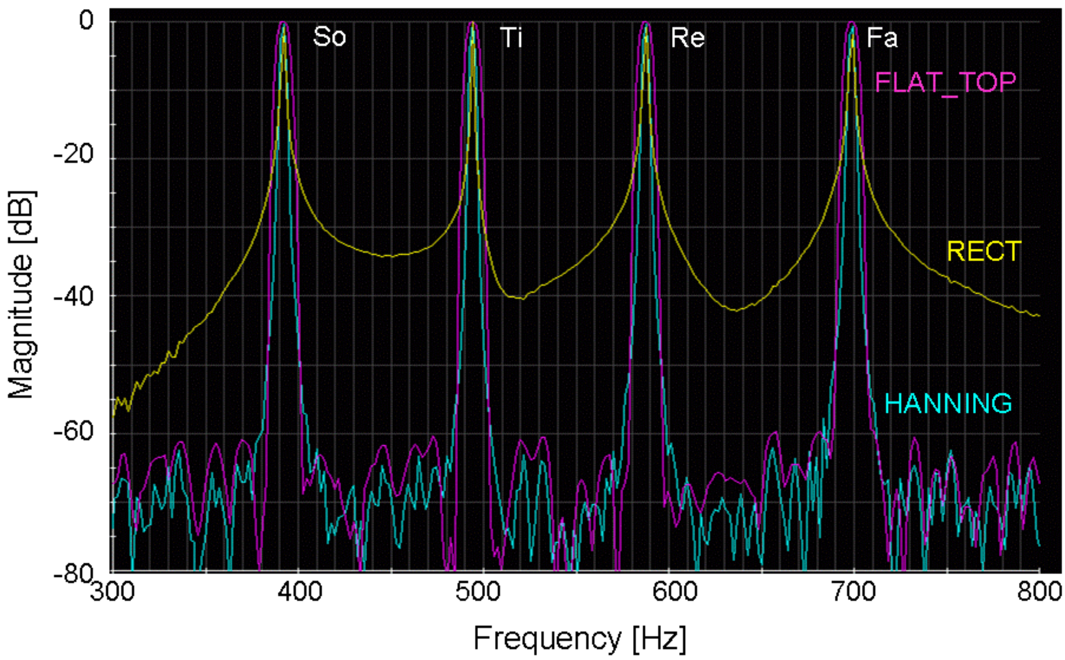


Figure 5: G7 Spectrum with 3 Windows

Processing the waveform by FFT, the frequency spectrum is shown as Figure 5. Yellow line (RECT) shows the case of no window. Each tone is not coherent so that four spectral lines are extremely smeared. FLAT_TOP and HANNING windows improve spectrum appearances.

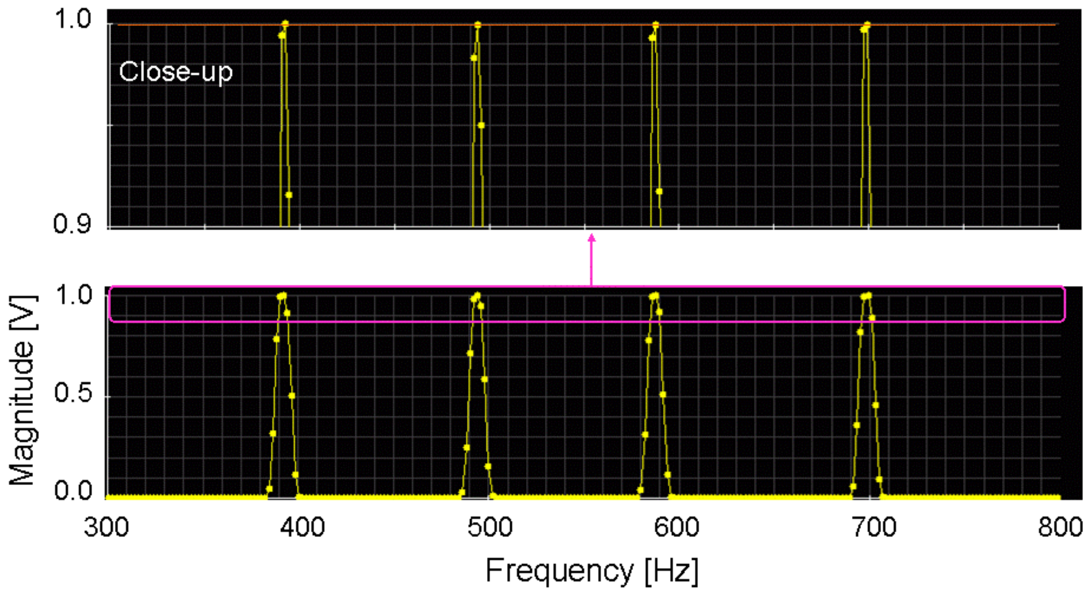


Figure 6: Spectrum with FLAT_TOP (Linear Y-Axis)

Figure 6 shows the linear scale amplitude spectrum with FLAT_TOP. The four peaks meet the line of 1.0 respectively. The read-out values of the frequencies and amplitudes are listed in Table 1. FLAT_TOP can accurately estimate the amplitudes.

Figure 7 shows the linear scale amplitude spectrum with HANNING. The local maximums and the 2nd maximums at each peak are high-lighted with white and orange circles respectively.

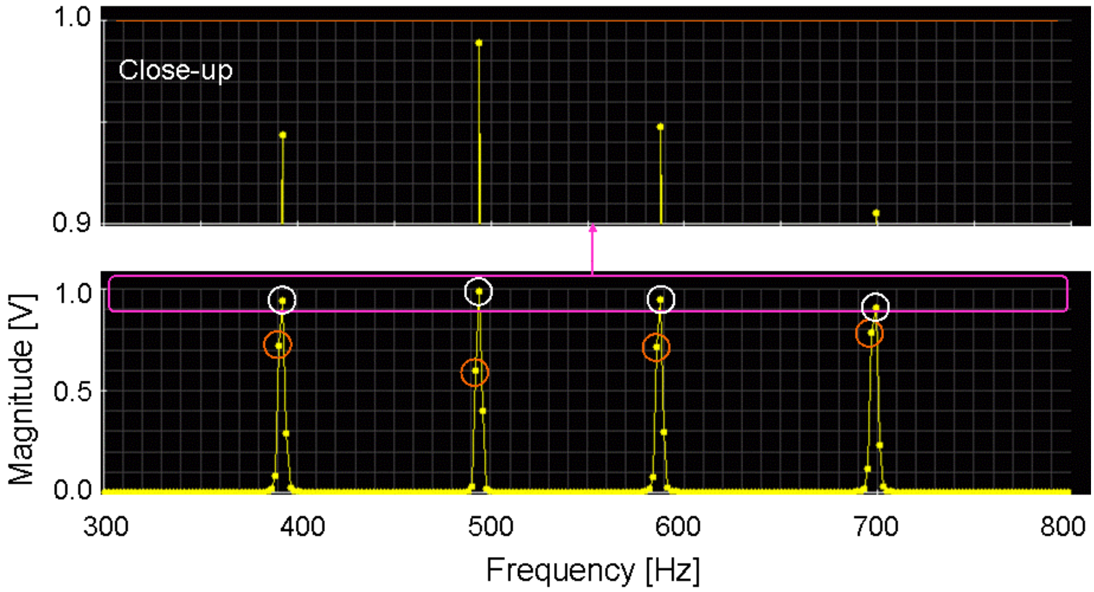


Figure 7: Spectrum with HANNING (Linear Y-Axis)

When HANNING window, you can apply the Tabei & Ueda method discussed previously to each combination, and the estimated frequencies and amplitudes are listed in Table 1 as well. The frequencies and the amplitudes are precisely estimated.

	True Freq.	FLAT_TOP		HANNING + T&U	
		Freq. Error	Amp. Est.	Freq. Error	Amp. Est.
	Hz	Hz	V	Hz	V
So	392.00	0.5827	1.000000	-0.00051	1.00005
Ti	493.88	0.2573	0.999295	0.00042	0.99979
Re	587.33	0.5611	0.999527	-0.00013	0.99976
Fa	698.46	0.7623	0.999536	-0.00042	0.99977

Table 1: Estimation Results

Accuracy Test

Let's see how accurately the method can estimate the frequency, amplitude and phase. This is a simulation that a 1V single tone waveform is digitized at the rate of 110Msps and captured 1024 points. The signal contains random noise of S/N=48dB. The phase offset is programmed $\pi/4$. When 96.4 cycles is captured in the unit test period, its spectrum looks as Figure 8.

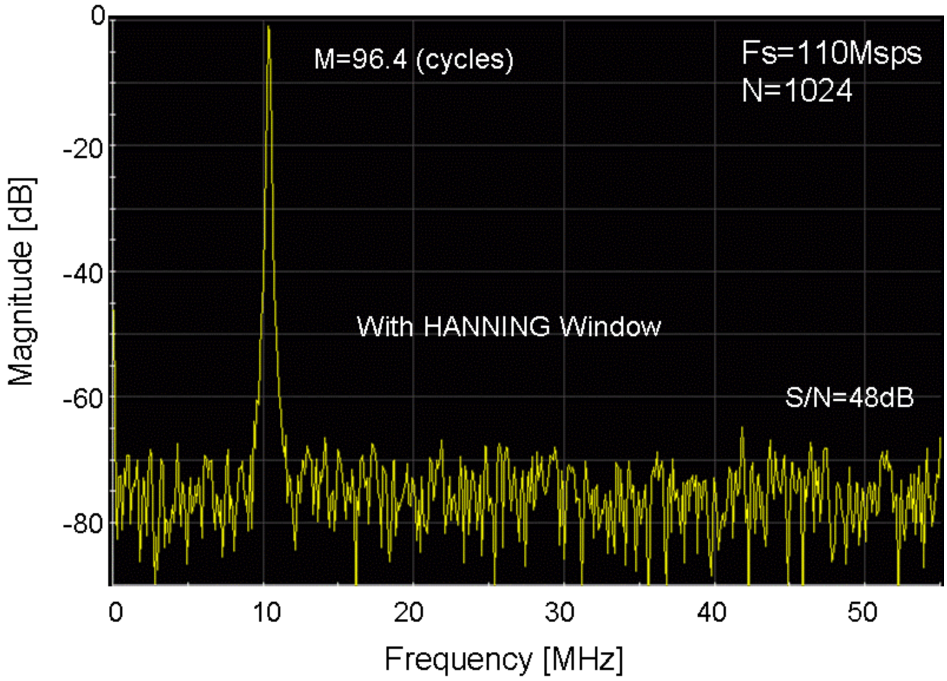


Figure 8: Smeared Spectrum

The test frequency is swept from approximately 200kHz to 54MHz with 100kHz step. The Tabei & Ueda method is performed to each one of them, and the estimation error of the frequency, amplitude and phase shown in Figures 9, 10 and 11 respectively.

Except near the DC and the Nyquist frequency, the frequency error is approximately +/-60Hz, the amplitude error is approximately +/-0.0006V and the phase error is approximately +/-0.002rad. Considering the simulation is performed under the condition of S/N=48dB, this method really works well.

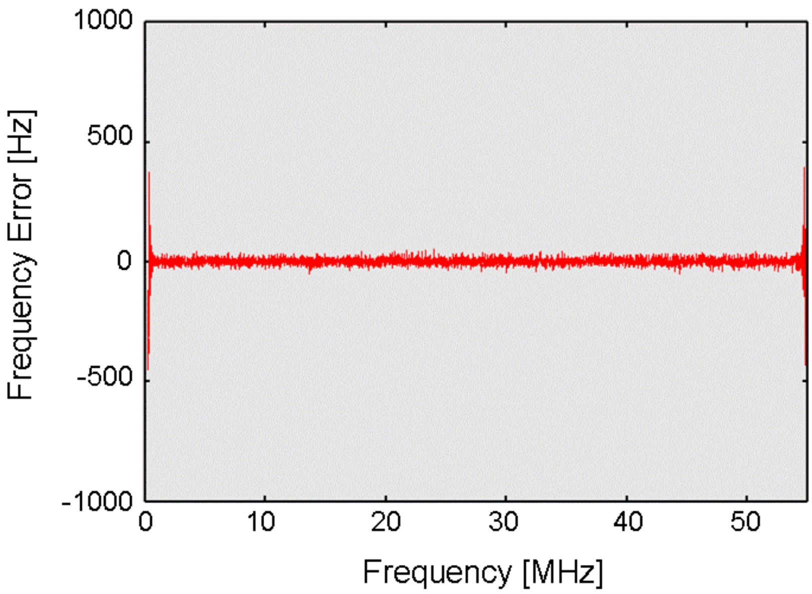


Figure 9: Frequency Error

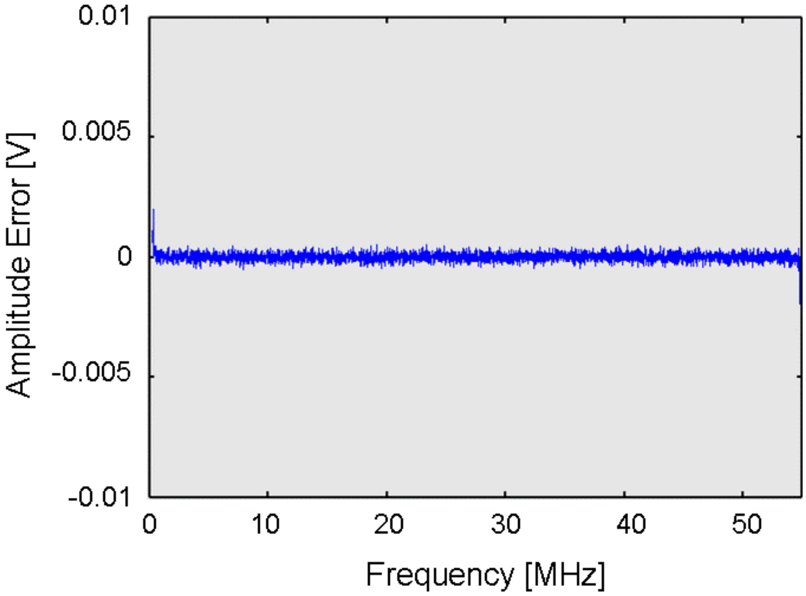


Figure 10: Amplitude Error

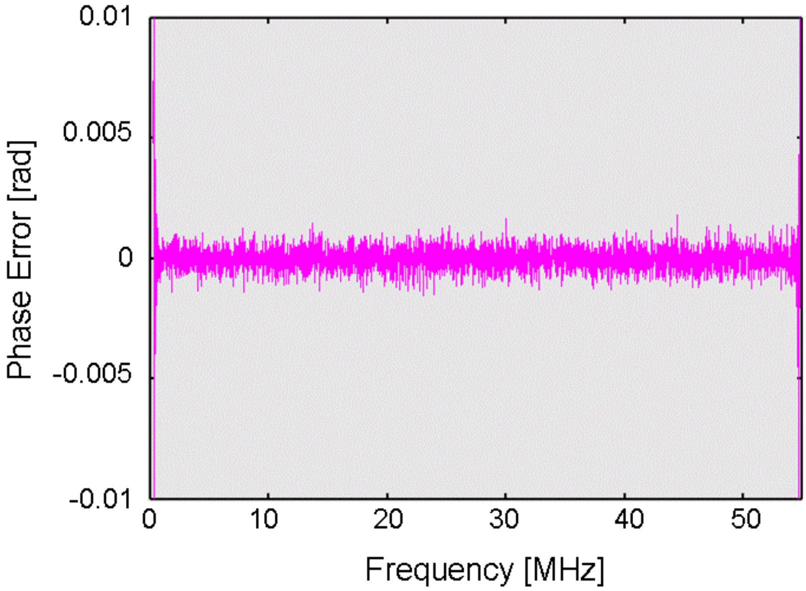


Figure 11: Phase Error

Conclusion

As you see the simulation results, the Tabei & Ueda interpolation method is really powerful so that it would be very useful in our mixed signal tests. For some reason, if you cannot settle your test condition with an integer number of cycles, you can apply this method and retrieve precise parameter values.